Nonlinear analysis of piles in rock
Analyse non-linéaire des pieux en roche

J. M. AMIR, Consulting Engineer, Tel-Aviv, Israel

SYNOPSIS

The behavior of shear piles in rock is analyzed by the spring model method, assuming an exponential relationship between sidewall shear and displacement. The resulting nonlinear differential equation, in terms of dimensionless force, may be solved by iterative finite differences. The load-settlement curves and axial force distribution obtained from this solution show good agreement with field measurements.

INTRODUCTION

Since the early seventies, drilled cast-in-situ piles have become the most important foundation method in the rocky regions of Israel. The main reason for this is that piling, especially in jointed and karstic rock, has obvious economical advantages over shallow footings (Amir 1983). The rapid advance in construction techniques has, however, left analytical techniques behind. In order to achieve safer and more economical design, engineers need better understanding of the way these piles function. The paper presented here proposes a new analytical approach in this direction.

SIDEWALL SHEAR MODELLING

Piles in rock (with the exception of short, large-diameter sockets) derive their capacity mainly from sidewall shear. The problem, therefore, is reduced to modelling the behavior of the pile in shear. Basically, there are three analytical techniques which can be used to model the load-deformation behavior of piles:

a) The spring model (Scott 1981)

b) The half-space model (Hettes & Poulos 1969)

c) The finite element method (Pells & Turner 1979).

The spring model was chosen for this work because of its relative simplicity, and because the extra computational effort involved with the other (more accurate) methods can only be justified if reliable material parameters are available. Unfortunately, rock testing techniques have not yet matured to this stage.

In its simplest form, the spring model is linear, and may be characterized by a single spring element. Although it can represent the behavior of piles under relatively small loads, this single-parameter model can simulate neither work-hardening nor yield.

The elasto-plastic model, which consists of a spring and a friction element connected in series, does represent yield phenomena. Still, it lacks continuity and is grossly in error in the working range of the pile, which is of most interest to the engineer.

Multi-element elasto-plastic models, consisting of a series of springs and friction elements, have none of the above shortcomings. Generally, the stress-displacement curves for such models are in the shape of broken lines. In the special case where both spring and friction values decrease in a geometrical progression (Fig. 1), the slopes of the successive sections also form a descending geometrical progression. In the limit, if an infinite number of elements is taken, a smooth curve, obeying Eq. (1), results:

\[
\ln (r') = s - n.s
\]

where \(r'\) is the sidewall shear stress, \(s\) the displacement and both \(s\) and \(n\) are constants. Rearranging and integrating, Eq. (1) becomes:

![Fig. 1 - Multi-element elasto-plastic model](image-url)
\[
\tau = K - e^a/n
\]

(2)

Since \( \tau = 0 \) when \( a = 0 \), the integration constant \( K \)

is equal to \( e^a/n \), therefore:

\[
\tau = m(1-e^{-n \lambda})
\]

(3)

where \( m \), the yield shear stress, is equal to \( e^a/n \). This

exponential expression has the required attributes of
strain-hardening, continuity and eventual yield, still

using only two parameters. Eq. (3) is identical to the

expression suggested by Scott (1981), and to the load-

settlement function for complete piles given by van der

Veen (1953). By differentiating at the origin, it can be

verified that \( mn \) is the corresponding tangent

modulus.

For the linearly elastic case, it can be shown

(Appendix I), that the pile modulus (the ratio of mean

side wall shear to pile settlement) is practically inde-

pendent of the pile diameter. It was therefore assumed

that, for given site conditions, the spring parameters

\( m \) and \( n \) are constant.

Unfortunately, there is little published data regarding

shear tests between concrete and rock. Results of such

tests, reported by Pells et al. (1980) and Williams &

Pells (1981), show that the exponential dependence of

shear stress on displacement is basically correct.

LOAD-SETTLEMENT BEHAVIOR

Under an applied axial load \( Q \), the pile deforms

with depth \( z \) according to the function \( s = s(z) \). For a

pile section with a length \( l \) and circumference \( C \), the

axial force changes from \( F \) to \( F + AF \), and the equilib-

rium condition gives:

\[
F' = \frac{dF}{dz} = \frac{s}{z} - \tau \cdot l \cdot C
\]

substituting the value of \( \tau \) from Eq. (3), one gets:

\[
F' = C \cdot m(1 - e^{-n \lambda})
\]

(5)

differentiating,

\[
F'' = C \cdot m \cdot n \cdot e^{-n \lambda} \cdot s'
\]

(6)

Hooke's law for the same pile section yields:

\[
s' = -\frac{F}{E A}
\]

(7)

where \( E \) and \( A \) are the modulus of elasticity and the

cross-sectional area of the pile, respectively. Substitu-
ting in (6):

\[
F'' = C \cdot m \cdot n \cdot e^{-n \lambda} \cdot \frac{F}{E A}
\]

(8)

Combining (5) and (8) yields:

\[
F' = \frac{n}{\lambda \cdot E A} - \frac{n \cdot C \cdot m}{E A}
\]

(9)

Equation (9) may be normalized by the following substi-
tutions:

\[
\frac{n}{\lambda \cdot E A} = F
\]

(10)

and

\[
\frac{C \cdot m}{E A} = Q
\]

(11)

so that Eq. (9) becomes:

\[
F' = \frac{n \cdot Q}{C \cdot m n} \cdot \frac{C \cdot m n}{\lambda \cdot E A}
\]

(12)

the derivatives being with respect to the dimensionless

depth \( \xi \). To simplify Eq. (12), it is convenient to

make the following substitutions:

\[
\frac{C \cdot m n}{\lambda \cdot E A} = \frac{m \cdot C \cdot E \cdot A}{Q \cdot n}
\]

(13)

thus:

\[
\frac{n}{C \cdot m n} \cdot \frac{n \cdot Q}{\lambda \cdot E A} = \frac{n}{C \cdot m n} \cdot \frac{n \cdot Q}{\lambda \cdot E A} = k \cdot \vartheta
\]

(15)

Eq. (15) is a non-linear differential equation in terms of

the dimensionless force \( \vartheta \). This is a convenient

form, since in practice the boundary values are also

known in terms of force: At the top \( \vartheta = 0 \), so \( \vartheta \)

is equal to unity. As for the bottom \( \vartheta = 1 \), it was

demonstrated (Mates & Poulos 1969) that for normal

sledderness ratios the proportion of the force reaching

the bottom does not exceed a few percent. For shear

piles in rock, in which debris is allowed to collect at

the bottom, assuming zero force \( (\vartheta = 0) \) at the end will

usually be quite appropriate. For extremely long piles,

it may be more accurate to assume that the point of

zero axial force is higher up, but this does not intro-

duce any additional difficulty.

Eq. (15) has no closed-form analytical solution, and a

conventional finite difference formulation produces non-linear algebraic equations which are difficult to

solve. To overcome this difficulty, a fictitious sec-

tion was added on top of the pile, and an arbitrary

value of \( \vartheta \) assumed at the top of this fictitious

section. Using central differences then produced an

algorithm that yielded the value of \( \vartheta \) at the next

node. This procedure was repeated until the value of \( \vartheta \)

at the bottom has been obtained. Depending upon the

sign of this, the value of \( \vartheta \) at the top of the ficti-

tious section was iteratively adjusted until a value of

\( \vartheta \) close enough to zero was reached at the bottom.
the distribution of forces along the pile is shown, the corresponding distribution of sidewall shear stresses is easily obtained, enabling the calculation of settlements from Eq. (3). An important precaution, however, is not to base the settlement calculation on shear stress on the top section of the pile: At the top, the shear stress can be very close to the yield stress, and this can introduce a large numerical error. Instead, the settlement was computed from the shear at the bottom. The shortening of the pile, calculated by integrating \( s' \) from Eq. (7), was then added to the settlement of the top.

The procedure described above, The load-settlement behavior for piles of different diameters and lengths predicted, as well as the distribution of axial force for any given load.

**Comparison with Experimental Data**

**Caissons in Mica Schist - Philadelphia**

A high-rise building in Philadelphia was underpinned on instrumented caissons socketed deep into sound mica schist (Koutsoufas 1981). Since the sockets were designed with a very high factor of safety, the load-settlement curves were essentially linear. Using the yield stress \( m \) quoted by the author (1500 KPa), a value of \( n = 2900 \text{ m}^{-1} \) was back-calculated from the load-settlement curve given for a 610 mm diameter socket. The load distribution along the pile was then computed for two different loads, comparing well with the measured values (Fig. 3).

**Piles in Mudstone - Melbourne**

Four instrumented test piles were installed in moderately weathered mudstone and tested by Williams et al. (1980). Test pile No. M10 (diameter 660 mm, length 7.8 m) settled 3.6 mm under a load of 7550 KN. Based on the back-calculated parameters \( m = 900 \text{ KPa} \) and \( n = 2900 \text{ m}^{-1} \), the load distribution along pile M10 was calculated for an applied load of 7650 KN. Again the results (Fig. 4) show a marked resemblance to the measured values.
CONCLUSIONS

a. The dependence of sidewall shear on displacement may be represented by an exponential function, with the yield stress $m$ and the tangent modulus $m_n$ as parameters.

b. Using this function leads to a nonlinear differential equation in terms of force, describing the complete behavior of shear piles in rock.

c. The parameters $m$ and $n$ may be obtained directly from shear tests, either in the field or in the laboratory. In uniform rock, $m$ and $n$ may also be back-calculated from the results of pile load tests.

d. Using the parameters thus obtained, the prediction of load distribution along the pile becomes rather straightforward, and the results show good resemblance to in-situ distributions measured in a variety of rocks.

APPENDIX I - INFLUENCE OF THE DIAMETER ON THE SETTLEMENT OF PILES

According to Matese & Poulos (1969), the settlement of piles in a linearly elastic half-space is given by:

$$ s = \frac{Q}{1 \cdot E_R} \quad (16) $$

where $I_p$ is an influence factor and $E_R$ the rock mass modulus. For modulus ratios $K$ which are typical of rock, the $I_p$ values published by Matese & Poulos are roughly in direct proportion to the slenderness ratio $1/D$ (Fig. 5). Therefore, $I_p$ may be approximately substituted by:

$$ I_p = \frac{1}{B} \quad (17) $$

where $B = B(K)$ is constant for a given modulus ratio. Substituting in Eq. (16) results:

$$ s = \frac{Q \cdot B^1}{E_R} \quad (18) $$

where $F$ is the mean sidewall shear. The pile modulus $M_p$ is, therefore:

$$ M_p = \frac{F}{s} = \frac{E_R}{B} \quad (19) $$

From Eq. (19) it emerges that, at least as a first approximation, the pile modulus is independent of the pile diameter $D$.

REFERENCES


Fig. 5 - Influence factor $I_p$ vs. slenderness ratio $1/D$